

Low Energy Dynamics of N=2 Supersymmetric Monopoles

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Abstract

It is argued that the low-energy dynamics of k monopoles in N=2 supersymmetric Yang-Mills theory are determined by an N=4 supersymmetric quantum mechanics based on the moduli space of k static monopole solutions. This generalises Manton's "geodesic approximation" for studying the low-energy dynamics of (bosonic) BPS monopoles. We discuss some aspects of the quantisation and in particular argue that Dolbeault cohomology classes of the moduli space are related to bound states of the full quantum field theory.

1. Introduction

A significant breakthrough in our understanding of the interactions of monopoles followed the work of Manton [1]. He suggested a method of approximating the low-energy interactions of monopoles in Yang-Mills Higgs theory in the BPS limit. In this limit there is a Bogomol'nyi bound [2] on the static energy functional which implies the existence of static multi-monopole solutions that saturate the bound. The moduli space of static k -monopole solutions, \mathcal{M}_k , is a finite dimensional subspace of the configuration space that possesses a natural metric coming from the kinetic energy term in the Lagrangian. The low-energy dynamics of k monopoles is approximated by assuming that it is geodesic motion on \mathcal{M}_k . The metric on the moduli space for two monopoles was constructed by Atiyah and Hitchin [3] and has been used to develop both the classical and quantum scattering problems (see [4-6] and references therein).

Manton's approach can be generalised to study the low-energy dynamics of "solitons" of other systems including Abelian Higgs vortices [7], "lumps" of non-linear sigma models [8] and extremal black holes [9]. For this to work it is crucial that the theory admits static multi-soliton solutions. Generically, this means that Bogomol'nyi type bounds exist and that the theory has a supersymmetric extension.

It is thus natural to extend these considerations to study the low-energy interactions of solitons in supersymmetric theories. Due to the presence of fermions one is now necessarily considering the low-energy *quantum* theory. In a recent paper [10] (see also [11]), we initiated this extension in the context of the lump solutions of the $N = 2$ supersymmetric non-linear sigma model in $d = 2 + 1$. It was shown that the low-energy dynamics of k lumps are described by an $N = 2$ supersymmetric quantum mechanics based on the moduli space of k static lump solutions. In the present work we continue these investigations by studying the dynamics of monopoles of $N = 2$ supersymmetric Yang-Mills theory, the $N = 2$ extension of the BPS system¹.

We begin in section 2 with a review of BPS monopoles in the bosonic theory. The geometry of the moduli space of solutions is described and some notation is introduced that will be important in the later sections. The low-energy dynamics of monopoles in $N = 2$ supersymmetric Yang-Mills theory are discussed in section 3 and we show that they are determined by an $N = 4$ supersymmetric quantum mechanics (four real worldline supersymmetries) based on the moduli space of static BPS monopoles. In section 4 we discuss some aspects of the quantum theory and section 5 concludes with some discussion.

2. Review of BPS monopoles

2.1. Bogomol'nyi equations

Let A_m be an $SO(3)$ connection and Φ be a Higgs field transforming in the adjoint representation of $SO(3)$ governed by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}\text{Tr}F_{mn}F^{mn} + \frac{1}{2}\text{Tr}D_m\Phi D^m\Phi \quad (2.1)$$

¹ In an interesting recent paper [12] the (non-supersymmetric) quantum dynamics of iso-spinor fermions coupled to the BPS system is studied.

where F_{mn} is the curvature of the connection and $D_m = \partial_m + [A_m, \]$ is the covariant derivative. Since there is no potential term for the Higgs field, this being the BPS limit, spontaneous symmetry breaking of $SO(3)$ down to $U(1)$ is imposed by demanding that $\text{Tr}\Phi^2 = 1$ as a boundary condition at infinity ².

It will be convenient to work in the $A_0 = 0$ gauge. It is then necessary to impose Gauss' Law, the A_0 equation of motion, as a constraint on the time dependent physical fields:

$$D_i \dot{A}_i + [\Phi, \dot{\Phi}] = 0. \quad (2.2)$$

In this gauge the Lagrangian is given by $L = T - V$ where the kinetic energy T is given by

$$T = \frac{1}{2} \int d^3x \text{Tr}(\dot{A}_i \dot{A}_i + \dot{\Phi} \dot{\Phi}) \quad (2.3)$$

and the potential energy V takes the form

$$V = \frac{1}{2} \int d^3x \text{Tr}(B_i B_i + D_i \Phi D_i \Phi), \quad (2.4)$$

where $B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}$ is the non-abelian magnetic field strength. The total conserved energy is given by $E = T + V$.

In order to construct the static monopole solutions we need to minimise the static energy functional $E = V$. It was shown by Bogomol'nyi [2] that V can be rewritten as

$$V = \int d^3x \text{Tr} \left[\frac{1}{2} (B_i \mp D_i \Phi)(B_i \mp D_i \Phi) \right] \pm 4\pi k, \quad (2.5)$$

where

$$k = \frac{1}{4\pi} \int d^3x \partial_i \text{Tr}(B_i \Phi) \quad (2.6)$$

is the monopole number which, if the fields are smooth, is an integer topological invariant. We thus deduce the Bogomol'nyi bound

$$V \geq 4\pi |k|. \quad (2.7)$$

In each monopole class the static energy is minimised when the bound is saturated which is equivalent to the Bogomol'nyi equations

$$B_i = \pm D_i \Phi. \quad (2.8)$$

The upper sign corresponds to positive k or “monopoles” and the lower sign corresponds to negative k or “anti monopoles”. From now on we will restrict our considerations to monopoles, the extension to anti monopoles being trivial. From (2.7) we see that the total energy of k static monopoles is $4\pi k$. As is well known there are also electrically charged

² Our choice of units is the same as in [5].

dyons in this model. In the $A_0 = 0$ gauge there are no static dyon solutions; the dyons emerge as time dependent solutions (see e.g. [5] for more details).

It will be useful in the following to recall the well known fact that the Bogomol'nyi equations are equivalent to the self-duality equations of pure Yang-Mills in R^4 , restricted to be translationally invariant in one direction [13]. Specifically, we define a connection W_μ on R^4 that is translationally invariant in the x^4 direction via

$$W_i = A_i, \quad W_4 = \Phi. \quad (2.9)$$

If $G_{\mu\nu}$ is the field strength corresponding to W_μ , then the self duality equations for W_μ ,

$$G_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma}, \quad (2.10)$$

are equivalent to the Bogomol'nyi equations (2.8) (upper sign). Introducing the covariant derivative on R^4 , $D_\mu = \partial_\mu + [W_\mu, \cdot]$, Gauss' Law can be rephrased in terms of the Euclidean connection as

$$D_\mu \dot{W}_\mu = 0. \quad (2.11)$$

Finally an infinitesimal gauge transformations on (A_i, Φ) can be recast in the form

$$\delta W_\mu(x) = D_\mu \Lambda \quad (2.12)$$

if the gauge parameter $\Lambda(x)$ is restricted to be independent of x^4 .

2.2. Moduli space of BPS monopoles

Denoting the space of finite energy field configurations $\{W_\mu(x)\}$ by \mathcal{A} and the group of gauge transformations by \mathcal{G} , the configuration space of the BPS system is given by the quotient $\mathcal{C} = \mathcal{A}/\mathcal{G}$. That is, configurations related by gauge transformations are identified. Letting \dot{W} and \dot{V} be two tangent vectors on \mathcal{A} , a natural metric on \mathcal{A} is induced from that on R^4 via

$$\mathcal{G}(\dot{W}, \dot{V}) = \int d^3x \text{Tr}(\dot{W}_\mu \dot{V}_\mu). \quad (2.13)$$

A tangent vector to \mathcal{C} must satisfy Gauss' Law. From (2.11) and (2.12) it is clear that tangent vectors to \mathcal{C} are orthogonal to the gauge orbits and hence that the metric \mathcal{G} is also well defined on \mathcal{C} . In fact (2.13) is just twice the kinetic energy functional (2.3). Since the potential energy (2.4) is gauge invariant, this confirms that the configuration space for (2.1) is indeed \mathcal{C} .

The moduli space of k -monopoles, $\mathcal{M}_k \subset \mathcal{C}$, is defined as the complete set of solutions to the Bogomol'nyi equations (2.8) within a given topological class k . A natural set of coordinates for \mathcal{M}_k are the arbitrary parameters or moduli $\{X^a, a = 1, \dots, \dim(\mathcal{M}_k)\}$ that determine the most general gauge equivalence class of solutions $[W_\mu(x, X)]$. In addition to Gauss' Law, tangent vectors to \mathcal{M}_k must also obey the linearised Bogomol'nyi equations

$$D_{[\mu} \dot{W}_{\nu]} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} D_\rho \dot{W}_\sigma. \quad (2.14)$$

There is a very close connection between zero modes in the fluctuations about a particular monopole background and tangent vectors to \mathcal{M}_k . Using the co-ordinates $\{X^a\}$, we can express an arbitrary tangent vector to \mathcal{M}_k as $\dot{W}_\mu = \dot{X}^a \delta_a W_\mu$ provided $\delta_a W_\mu$ satisfies

$$D_{[\mu} \delta_a W_{\nu]} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} D_\rho \delta_a W_\sigma \quad (2.15)$$

and

$$D_\mu \delta_a W_\mu = 0. \quad (2.16)$$

Equation (2.15) is precisely the statement that $\delta_a W_\mu$ is a zero mode and equation (2.16) says that it is orthogonal to gauge modes. A metric on \mathcal{M}_k is naturally obtained from the restriction of the metric \mathcal{G} (2.13) on \mathcal{C} . In the co-ordinates $\{X^a\}$ this is given by

$$\mathcal{G}_{ab} = \int d^3x \text{Tr}(\delta_a W_\mu \delta_b W_\mu). \quad (2.17)$$

For the BPS system all zero modes are normalisable (i.e. finite \mathcal{G}_{ab}) and there is a one to one correspondence between normalisable zero modes and moduli. Note that this is not always true (e.g. for the lumps of non-linear sigma models [8]).

Following closely the work in [11], we now turn to a description of the zero modes that will be important later. Let $W_\mu(x, X)$ be a family of BPS monopole configurations and $\epsilon_a(x, X)$ be a set of arbitrary gauge transformations, then both $\partial_a W_\mu$ and $D_\mu \epsilon_a$ satisfy (2.15), where $\partial_a = \frac{\partial}{\partial X^a}$. Zero modes about the configuration $W_\mu(x, X)$ may then be constructed as the linear combination,

$$\delta_a W_\mu = \partial_a W_\mu - D_\mu \epsilon_a, \quad (2.18)$$

by demanding that the gauge parameters are chosen to satisfy (2.16). The gauge parameters ϵ_a can be viewed as defining a natural connection on \mathcal{M}_k with covariant derivative

$$s_a = \partial_a + [\epsilon_a, \]. \quad (2.19)$$

The form of (2.18) then suggests that we should consider the zero modes as the mixed curvature components of a generalised connection $(W_\mu(x, X), \epsilon_a(x, X))$ defined on $R^4 \times \mathcal{M}_k$. That is

$$[s_a, D_\mu] = \delta_a W_\mu. \quad (2.20)$$

Using the definitions

$$\begin{aligned} G_{\mu\nu} &= [D_\mu, D_\nu] \\ \phi_{ab} &= [s_a, s_b] \end{aligned} \quad (2.21)$$

Jacobi identities can be used to show that

$$s_a G_{\mu\nu} = 2D_{[\mu} \delta_a W_{\nu]} \quad (2.22)$$

and

$$D_\mu \phi_{ab} = -2s_{[a} \delta_b W_\mu. \quad (2.23)$$

From (2.23) we further deduce that the curvature ϕ_{ab} can be expressed in the simple form

$$\phi_{ab} = 2(D_\mu D_\mu)^{-1}[\delta_a W_\nu, \delta_b W_\nu]. \quad (2.24)$$

The connection (W_μ, ϵ_a) with curvature components $(G_{\mu\nu}, \delta_a W_\mu, \phi_{ab})$ is in fact the connection on the “universal bundle” introduced by Atiyah and Singer [14]. The above identities can be used to cast the Christoffel connection associated to the metric (2.17) in the form:

$$\Gamma_{abc} = \mathcal{G}_{ad}\Gamma^d_{bc} = \int d^3x \text{Tr}(\delta_a W_\mu s_b \delta_c W_\mu) \quad (2.25)$$

The space of field configurations \mathcal{A} inherits three almost complex structures, $\tilde{J}^{(m)}$, from those on R^4 , $J^{(m)}$, via

$$[\tilde{J}^{(m)}\dot{W}]_\mu(x) = J^{(m)}_{\mu\nu}\dot{W}_\nu. \quad (2.26)$$

It is straightforward to show that $\tilde{J}^{(m)}$ satisfy the same algebra as $J^{(m)}$ i.e. $\tilde{J}^{(m)}\tilde{J}^{(n)} = -\mathbf{1}\delta^{mn} + \epsilon^{mnp}\tilde{J}^{(p)}$. Using the fact that each complex structure satisfies

$$\epsilon_{\mu\nu\rho\sigma} = -(J^{(m)}_{\mu\nu}J^{(m)}_{\rho\sigma} + J^{(m)}_{\mu\sigma}J^{(m)}_{\nu\rho} + J^{(m)}_{\mu\rho}J^{(m)}_{\sigma\nu}) \quad (2.27)$$

(no sum on m) we can show that if $\dot{W}_\mu(x)$ is a tangent vector to \mathcal{M}_k (satisfies (2.11) and (2.14)) then so is $J^{(m)}_{\mu\nu}\dot{W}_\nu$. Thus the almost complex structures on \mathcal{A} , $\tilde{J}^{(m)}$, descend to give almost complex structures on \mathcal{M}_k , $\mathcal{J}^{(m)}$. In the co-ordinates $\{X^a\}$ they take the form

$$\mathcal{J}^{(m)}_a{}^b = \mathcal{G}^{cb} \int d^3x \mathcal{J}^{(m)}_{\mu\nu} \text{Tr}(\delta_a W_\mu \delta_c W_\nu) \quad (2.28)$$

To calculate the algebra satisfied by the $\mathcal{J}^{(m)}$ in this representation one needs to use the fact that the zero modes $\delta_a W$ and the massive modes ³ ψ_i in the fluctuations about a BPS monopole configuration form a complete set of functions

$$\mathcal{G}^{ab}\delta_a W_\mu^\alpha(x)\delta_b W_\nu^\beta(y) + \sum_i \psi_{i\mu}^\alpha(x)\psi_{i\nu}^\beta(y) = \delta^{\alpha\beta}\delta_{\mu\nu}\delta^3(x-y) \quad (2.29)$$

where α, β are $SO(3)$ indices. This identity can also be used to show that

$$\mathcal{J}^{(m)}_a{}^b \delta_b W_\mu = -J^{(m)}_{\mu\nu} \delta_a W_\nu. \quad (2.30)$$

Using the formulae presented above we can also show that the torsion vanishes

$$\partial_{[a}\mathcal{J}^{(m)}_{bc]} = 0. \quad (2.31)$$

Although the discussion presented here is only schematic, it can be rigorously shown that M_k is indeed a hyperKähler manifold [4].

³ Since the spectrum is continuous i is a continuous parameter.

2.3. Geodesic approximation

Let $W_\mu(x)$ correspond to a static multi-monopole solution in the $A_0 = 0$ gauge. As we noted the fluctuations about this solution contain massive modes in addition to the zero modes. To have a well defined perturbation scheme, we need to introduce a collective co-ordinate for each zero mode; these are the moduli X^a . We then expand an arbitrary time dependent field as a sum of massive modes with time dependent coefficients, by properly taking into account the time dependence of the collective co-ordinates (see, e.g. [15]). A low-energy ansatz for the fields is obtained by ignoring the massive modes and demanding that the only time dependence is via the collective co-ordinates. Thus we are led to the ansatz

$$\begin{aligned} W_\mu(x, t) &= W_\mu(x, X(t)) \\ A_0 &= \dot{X}^a \epsilon_a \end{aligned} \tag{2.32}$$

After substituting this into the action we obtain an effective action which is precisely that of a free particle propagating on the moduli space with metric (2.13)[1]. The equations of motion are simply the geodesics on the moduli space.

The A_0 term in (2.32) is necessary to ensure that the motion is constrained to be orthogonal to gauge transformations. It can be justified more formally as follows. Taking the dimension of X^a to be zero, the collective co-ordinate expansion is an expansion in the number of time derivatives $n = n_{\partial_t}$. Since the effective action is order $n = 2$ it is necessary to ensure that the equations of motion are satisfied to order $n = 0$ and $n = 1$. To order $n = 0$ this is true by assumption that the collective co-ordinates are the moduli parameters. The order $n = 1$ equation of motion is simply Gauss' Law and we have explicitly solved this equation for A_0 . Since we are working in the $A_0 = 0$ gauge we should really perform a gauge transformation on the ansatz to stay in this gauge. Of course this won't affect the result. We have chosen to write the ansatz in this unorthodox way since we construct a similar ansatz in the supersymmetric case.

3. N=2 supersymmetric monopoles

3.1. Monopole solutions

The N=2 supersymmetric extension of the Yang-Mills Higgs model (2.1) is described by the Lagrangian density

$$\begin{aligned} \mathcal{L} = \text{Tr} \{ & -\frac{1}{4}(F_{mn})^2 + \frac{1}{2}(D_m P)^2 + \frac{1}{2}(D_m S)^2 - \frac{1}{2}([S, P])^2 \\ & + i\bar{\chi}\gamma^m D_m \chi - \bar{\chi}\gamma_5[P, \chi] - i\bar{\chi}[S, \chi] \} \end{aligned} \tag{3.1}$$

where A_m is an $SO(3)$ connection and the Higgs fields P, S and the Dirac fermions χ are in the adjoint representation of the $SO(3)$. We choose a mostly minus metric and define $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. The supersymmetry transformations are given by

$$\begin{aligned} \delta A_m &= i\bar{\alpha}\gamma_m \chi - i\bar{\chi}\gamma_m \alpha \\ \delta P &= \bar{\alpha}\gamma_5 \chi - \bar{\chi}\gamma_5 \alpha, \quad \delta S = i\bar{\alpha}\chi - i\bar{\chi}\alpha \\ \delta \chi &= (\sigma^{mn} F_{mn} - \not{D}S + i\not{D}P\gamma_5 - i[P, S]\gamma_5) \alpha \end{aligned} \tag{3.2}$$

where α is a constant anticommuting Dirac spinor. The Lagrangian can be obtained by the dimensional reduction of supersymmetric Yang-Mills theory in six dimensions [16]. The Lagrangian is also invariant under chiral rotations which are a residuum of the rotation group in the extra dimensions.

Since the potential for the Higgs fields has flat directions corresponding to $[S, P] = 0$, the vacuum configuration is undetermined. We again impose spontaneous symmetry breaking by demanding $\text{Tr}S^2 + \text{Tr}P^2 = 1$ as a boundary condition at infinity. For S and P to commute in $SO(3)$ they must be proportional. By performing a chiral rotation we can assume that only S has a vacuum expectation value.

By straightforward calculation one finds that the supersymmetry algebra contains central charges [17]:

$$\{\bar{Q}, Q\} = 2\gamma_\mu P^\mu - 2Q_E - 2i\gamma_5 Q_M \quad (3.3)$$

where the electric and magnetic charges are given by

$$Q_E = \int d^3x \partial_i \text{Tr}(E_i \Phi) \quad Q_M = \int d^3x \partial_i \text{Tr}(B_i \Phi) = 4\pi k \quad (3.4)$$

respectively. The electric charges can be understood as arising from the components of the six dimensional momentum in the extra dimensions whilst the magnetic charges have a purely topological origin. The presence of the central charges in the algebra implies that there is a bound on the mass:

$$M^2 \geq Q_E^2 + Q_M^2. \quad (3.5)$$

In the vacuum sector, the elementary particles form a supermultiplet with charges and masses that saturate the bound.

To discuss the soliton sectors we again work with the $A_0 = 0$ gauge. In this gauge any static configuration has zero electric charge and the bound (3.5) is precisely the Bogomoln'yi bound we discussed in the bosonic case (2.7). Thus, just as in the bosonic case, the bosonic fields are partitioned into different topological sectors and for a given topological sector the static energy is minimised by solving the Bogomol'nyi equations

$$\begin{aligned} \frac{1}{2}\epsilon_{ijk}F_{jk} &= D_i \Phi \\ S &= \Phi \\ P = A_0 = \chi &= 0 \end{aligned} \quad (3.6)$$

These static monopole solutions break half of the supersymmetries. By this we mean that only half of the supersymmetry parameters in (3.2) will leave a solution of (3.6) invariant. Specifically, by introducing the following projection matrix

$$\Gamma_5 = -i\gamma_0\gamma_5, \quad \Gamma_5^2 = 1, \quad \Gamma_5^\dagger = \Gamma_5 \quad (3.7)$$

and defining

$$\alpha_\pm = \frac{1 \pm \Gamma_5}{2}\alpha \quad (3.8)$$

one can show that α_+ are the parameters of the unbroken supersymmetry and α_- are the parameters of the broken supersymmetry. This partial breaking of supersymmetry can also be deduced directly from the supersymmetry algebra (3.3) when the bound (3.5) is saturated.

3.2. Zero modes and supersymmetry

We now turn to the construction of the zero modes in the fluctuations about the monopole solution. The bosonic zero modes are exactly the same as those for the purely bosonic theory and we discussed them in the last section. The fermionic zero modes are time independent c-number solutions to the Dirac equation in the presence of the monopole:

$$i\gamma^i D_i \chi - i[\Phi, \chi] = 0. \quad (3.9)$$

It is straightforward to show that the broken supersymmetry with c-number parameters generates two independent zero modes satisfying $\Gamma_5 \chi = -\chi$. These are the fermionic Goldstone modes corresponding to the broken supersymmetry. In the one monopole sector these are the only fermionic zero modes.

In the multi-monopole sectors there are other fermionic zero modes; the Callias index theorem [18] states that in the k monopole sector there are $2k$ fermionic zero modes. Following closely an argument of Zumino's in the context of instantons [19], we now show how the bosonic and fermionic zero modes form a supermultiplet with respect to the unbroken supersymmetry. To exhibit this connection we first introduce some hermitean euclidean gamma matrices via

$$\Gamma_i = \gamma_0 \gamma_i, \quad \Gamma_4 = \gamma_0 \quad (3.10)$$

satisfying

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu} \quad (3.11)$$

and we note that $\Gamma_5 = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4$ justifying the notation introduced in (3.7). We next impose the following supersymmetric invariant restrictions on the equations of motion

$$\begin{aligned} \Gamma_5 \chi &= -\chi \\ A_0 &= -P \\ G_{\mu\nu} &= \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma} \end{aligned} \quad (3.12)$$

and demand that all fields are time independent.

The equations of motion then read

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma} \\ \Gamma_\mu D_\mu \chi &= 0 \\ D_\mu D_\mu P &= i[\chi^\dagger, \chi] \end{aligned} \quad (3.13)$$

and the unbroken supersymmetry transformations are given by

$$\begin{aligned} \delta W_\mu &= i\alpha_+^\dagger \Gamma_\mu \chi - i\chi^\dagger \Gamma_\mu \alpha_+ \\ \delta \chi &= -2\Gamma_\mu D_\mu P \alpha_+ \\ \delta P &= 0 \end{aligned} \quad (3.14)$$

The equations of motion (3.13) are covariant with respect to these transformations if α_+ is a Grassmann odd spinor. If $\alpha_+ = \epsilon_+$ is a c-number spinor, the Dirac equation in (3.13) is not covariant. However, if we impose the Dirac equation then the other two equations are covariant. Thus for each fermionic zero mode satisfying $\Gamma_5 \chi = -\chi$, we conclude that

$$\delta W_\mu = i\epsilon_+^\dagger \Gamma_\mu \chi - i\chi^\dagger \Gamma_\mu \epsilon_+ \quad (3.15)$$

is a bosonic zero mode; it can be further checked that it also satisfies (2.16). This seems to imply there are four bosonic zero modes for each fermionic zero mode, corresponding to the four real independent parameters ϵ_+ . However, they are not all independent: one can show that the four bosonic zero modes $(\delta_a W_\mu, J_{\mu\nu}^{(m)} \delta_a W_\nu)$ are paired with two fermionic zero modes $(\chi, C\chi^*)$ where $J_{\mu\nu}^{(m)}$ are the three complex structures on R^4 and the matrix C satisfies $C^2 = -1$, $C\Gamma_\mu = \Gamma_\mu^* C$.

Selecting a particular complex structure $J_{\mu\nu} \equiv -J_{\mu\nu}^{(3)}$ on R^4 it will be convenient to pair the bosonic and fermionic zero modes via [11]

$$\chi_a = \delta_a W_\mu \Gamma^\mu \epsilon_+ \quad (3.16)$$

where ϵ_+ is a c-number spinor satisfying

$$\begin{aligned} \epsilon_+^\dagger \epsilon_+ &= 1 \\ J_{\mu\nu} &= -i\epsilon_+^\dagger \Gamma_{\mu\nu} \epsilon_+ \\ J_{\mu\nu} \Gamma^\nu \epsilon_+ &= i\Gamma^\mu \epsilon_+ \end{aligned} \quad (3.17)$$

(the last equation is derivable using a Fierz identity from the second). Using (2.30) we deduce that the fermionic zero modes satisfy

$$\mathcal{J}_a^b \chi_b = i\chi_a \quad (3.18)$$

and hence that two bosonic zero modes are paired with one fermionic zero mode. The dimension of the moduli space of k static monopoles is $4k$ [4]. From the above analysis we conclude that there are $2k$ fermionic zero modes in the k monopole sector, in accord with the Callias index theorem [18].

3.3. Collective co-ordinate expansion

To analyse the dynamics of the monopoles we must introduce a collective co-ordinate for each zero mode. As we discussed in the last section, the bosonic collective co-ordinates are co-ordinates on moduli space. The fermionic collective co-ordinates λ^a are complex one component Grassmann odd objects. To construct a low-energy ansatz we ignore all non-zero modes and use the supersymmetric pairing of the zero modes (3.16).

First consider

$$\begin{aligned} W_\mu &= W_\mu(x, X(t)) \\ \chi &= \delta_a W_\mu \Gamma^\mu \epsilon_+ \lambda^a(t) \end{aligned} \quad (3.19)$$

with λ^a satisfying

$$-i\lambda^a \mathcal{J}_a{}^b = \lambda^b \quad (3.20)$$

due to (3.18). Denoting n_∂ as the number of time derivatives and n_f as the number of fermions, the collective co-ordinate expansion is an expansion in $n = n_\partial + \frac{1}{2}n_f$. The effective action obtained by substituting the ansatz (3.19) into the action is order $n = 2$. To have a consistent expansion we must ensure that the ansatz solves the equations of motion to order $n = 0, \frac{1}{2}, 1$. To order $n = 0, \frac{1}{2}$ (3.19) solves the equations of motion trivially. Just as in the bosonic case (2.32), to ensure the equations of motion are solved to order $n = 1$, it is necessary to supplement the naive ansatz (3.19) with

$$\begin{aligned} A_0 &= \dot{X}^a \epsilon_a - i\phi_{ab} \lambda^{\dagger a} \lambda^b \\ P &= i\phi_{ab} \lambda^{\dagger a} \lambda^b \end{aligned} \quad (3.21)$$

where ϵ_a and ϕ_{ab} were defined in the last section. The form of these terms is motivated by considering (3.13) and (2.24). Since we have been working in the $A_0 = 0$ gauge the ansatz (3.19)-(3.21) should really be gauged transformed into this gauge. However such an ansatz does not seem to be as simple to write down.

After substituting the ansatz into the action (3.1), integrating over the spatial degrees of freedom and using the various results of the last section we obtain

$$S_{\text{eff}} = \frac{1}{2} \int dt \mathcal{G}_{ab} \{ \dot{X}^a \dot{X}^b + 4i\lambda^{\dagger a} D_t \lambda^b \} - 4\pi k \quad (3.22)$$

where the covariant derivative is defined as

$$D_t \lambda^b = \dot{\lambda}^b + \Gamma_{ac}^b \dot{X}^a \lambda^c \quad (3.23)$$

and the Christoffel connection is given in (2.25). The constant term $4\pi k$ is simply the energy of the static k monopole configuration. Using complex co-ordinates (based on \mathcal{J}) it is convenient to redefine a real set of independent fermions ψ^a via

$$\psi^\alpha = \sqrt{2}\lambda^\alpha, \quad \psi^{\bar{\alpha}} = \sqrt{2}\lambda^{\dagger \bar{\alpha}} \quad (3.24)$$

(note that because of (3.20) $\lambda^{\bar{\alpha}} = \lambda^{\dagger \alpha} = 0$). The effective action can then be written in the form

$$S_{\text{eff}} = \frac{1}{2} \int dt \mathcal{G}_{ab} \{ \dot{X}^a \dot{X}^b + i\psi^a D_t \psi^b \} - 4\pi k \quad (3.25)$$

Since the metric on \mathcal{M}_k is hyperKähler, the effective action is invariant under $N = 4$ worldline supersymmetry:

$$\begin{aligned} \delta X^a &= i\beta_4 \psi^a + i\beta_m \psi^b \mathcal{J}_b^{(m)a} \\ \delta \psi^a &= -\dot{X}^a \beta_4 + \beta_m \dot{X}^b \mathcal{J}_b^{(m)a} \end{aligned} \quad (3.26)$$

where β_m, β_4 are four real Grassmann odd worldline parameters and $\mathcal{J}^{(m)}$ are the three complex structures on \mathcal{M}_k introduced in (2.28). Not surprisingly, the origin of these supersymmetries is precisely the unbroken supersymmetries of the field theory. The unbroken supersymmetries are parametrised by a four dimensional chiral spinor α_+ which has four independent real components. Writing

$$\begin{aligned}\delta W_\mu &= \delta X^a \delta_a W_\mu \\ \delta \chi &= \delta_a W_\mu \Gamma^\mu \epsilon_+ \delta \lambda^a + s_a (\delta W_\mu) \Gamma^\mu \epsilon_+ \lambda^a\end{aligned}\tag{3.27}$$

we substitute into the left hand side a supersymmetry transformation (3.2) with parameter $\sigma \epsilon_+$ or $\rho C \epsilon_+^*$, where σ and ρ are complex one component Grassmann odd objects and the matrix C was introduced after (3.15). After some lengthy algebra one obtains (to order $n = 1$) the transformations in (3.27) with β_1, β_2 and β_3, β_4 related to the real and imaginary parts of ρ and σ , respectively.

In conclusion the effective action governing the low-energy dynamics of the monopoles of N=2 supersymmetric Yang-Mills is given by an N=4 supersymmetric quantum mechanics based on the moduli space of static BPS monopoles.

4. Quantisation of the effective action

In the collective co-ordinate expansion we neglected all of the non-zero modes. In the quantum theory this is equivalent to demanding that all of these modes are in their ground states. In particular all radiative processes are neglected. The quantisation of the action (3.25) describes the low energy dynamics of k supersymmetric monopoles in this approximation. Determining the range of validity of this approximation seems a difficult problem; we will be content to assume that it is a reasonable approximation at low enough energies.

In [10] we showed that the effective action describing the dynamics of the lumps of the supersymmetric sigma model in $d = 2 + 1$ is also given by (3.25). For that case the moduli space is Kähler and hence the action has only $N = 2$ worldline supersymmetry. We discussed in detail in [10] the quantisation of (3.25) on a general Kähler moduli space and we refer the reader to that paper for more details on the following discussion.

For the one monopole sector the moduli space is given by $R^3 \times S^1$ (see e.g. [5]). The R^3 corresponds to the location of the monopole and the S^1 corresponds to the electric charge. In the bosonic case, the quantum Hamiltonian is given by

$$H = \frac{1}{8\pi} \mathbf{P}^2 + \frac{1}{8\pi} Q_E^2 + 4\pi\tag{4.1}$$

with $\mathbf{P} = -i\hbar \partial / \partial \mathbf{X}$, $Q_E = -i\hbar \partial / \partial \chi$ where \mathbf{X} , $0 \leq \chi \leq 2\pi$ are co-ordinates on R^3 and S^1 , respectively. Since χ is a periodic co-ordinate the electric charge is quantized; one obtains a tower of electrically charged dyon states with charge $Q_E = n\hbar$ and mass $4\pi + n^2 \hbar^2 / 8\pi$. This differs from the exact classical dyon mass formula [20] $\sqrt{16\pi^2 + n^2 \hbar^2} = \sqrt{Q_M^2 + Q_E^2}$ by an amount which is assumed to be small in the geodesic approximation (small electric charges).

We now outline the quantisation in the one monopole sector of the supersymmetric theory i.e. the quantisation of the effective action (3.25) on the target $R^3 \times S^1$. Using complex co-ordinates, the fermionic commutation relations $\{\psi^\alpha, \psi^{\bar{\beta}}\} = \hbar \delta^{\alpha\bar{\beta}}$ lead to a Hilbert space consisting of four types of states:

$$f_1(X)|0 > \quad f_{\bar{\alpha}}\psi^{\bar{\alpha}}|0 > \quad f_4(X)\psi^{\bar{1}}\psi^{\bar{2}}|0 > \quad (4.2)$$

where $\psi^\alpha|0 > \equiv 0$. The Hamiltonian acting on these states is given by (4.1). Thus each dyon state in the bosonic theory is now part of a degenerate dyon multiplet. The spins of these states can be calculated by constructing the semi-classical spin operator along the lines of [21]. It was noted in [21] that the multiplet consists of monopoles of spin $\frac{1}{2}$ and two with spin 0. In fact this multiplet provides a representation of the supersymmetry algebra (3.3) with the bound (3.5) saturated. Since the classical masses of all the particles in the spectrum (monopoles, dyons, elementary particles) saturate the bound and since the representations of (3.3) are larger when the bound is not saturated, Witten and Olive conjectured [17] that there are no quantum corrections to the classical masses.

The results of the previous section allow us to address the low-energy dynamics of multi monopole or dyon configurations. To do this we need to quantise the effective $N = 4$ supersymmetric quantum mechanics action (3.25) on the moduli space of k -monopoles, \mathcal{M}_k . In [10] we discussed in detail the quantisation of (3.25) on a general Kähler moduli space. It was noted there that for general Kähler manifolds there is an operator ordering ambiguity in the quantisation procedure; the Hilbert space of states is either anti-holomorphic forms or spinors on the moduli space. In the present case where the target manifold is hyperKähler and hence Ricci flat, these two quantisations are equivalent and so there is no ambiguity.

For definiteness we will work with anti-holomorphic forms, although for more detailed calculations it may be more convenient to use spinors. The Hilbert space of states is thus given by (superpositions of) anti-holomorphic differential forms on \mathcal{M}_k :

$$\begin{aligned} |f > &= \psi^{\bar{\alpha}_1} \dots \psi^{\bar{\alpha}_p} |0 > = \frac{1}{p!} f_{\bar{\alpha}_1 \dots \bar{\alpha}_p} \\ |f > &\leftrightarrow \frac{1}{p!} f_{\bar{\alpha}_1 \dots \bar{\alpha}_p} dZ^{\bar{\alpha}_1} \wedge \dots \wedge dZ^{\bar{\alpha}_p} \end{aligned} \quad (4.3)$$

where $\{Z^\alpha, Z^{\bar{\alpha}}\}$ are complex co-ordinates on \mathcal{M}_k and $\psi^\alpha|0 > = 0$. The degree of the differential form corresponds to the number of fermionic zero modes that are excited. The quantum hamiltonian is given by the Laplacian acting on differential forms

$$H = \hbar^2 (\bar{\partial}^\dagger \bar{\partial} + \bar{\partial} \bar{\partial}^\dagger) + 4\pi k = \frac{\hbar^2}{2} (dd^\dagger + d^\dagger d) + 4\pi k \quad (4.4)$$

where d is the exterior derivative, d^\dagger is its adjoint, $\bar{\partial}$ and $\bar{\partial}^\dagger$ are the anti-holomorphic analogues and the equality uses the Kähler property of the moduli space.

Asymptotically, \mathcal{M}_k is isomorphic to k copies of \mathcal{M}_1 thus providing the interpretation of k well separated monopoles. In this region an anti-holomorphic form with non-zero

asymptotic support can be written as (a sum of) wedge products of forms on each of the k copies of \mathcal{M}_1 :

$$|f\rangle \leftrightarrow f \approx f^{(1)} \wedge \dots \wedge f^{(k)} \quad (4.5)$$

where $f^{(i)}$ is a differential form on \mathcal{M}_1 (a state of the form (4.2)). The interpretation of these states is that of well separated monopoles with fermionic zero modes excited on each. They can thus be identified with (superpositions of) monopoles and dyons in the spectrum.

To proceed further we use the fact [4] that \mathcal{M}_k isometrically decomposes as

$$\mathcal{M}_k = R^3 \times \left(\frac{S^1 \times \mathcal{M}_k^0}{Z_k} \right). \quad (4.6)$$

The R^3 corresponds to the location of the centre of mass and the S^1 corresponds to the total electric charge of the k monopoles. The hyperKähler manifold \mathcal{M}_k^0 is the interesting part and determines the relative motion. The presence of the Z_k is a reflection of the fact that the monopoles are not classically distinguishable. First consider quantising on $\mathcal{M}_1 \times \mathcal{M}_k^0$. Since \mathcal{M}_k^0 is hyperKähler, the quantum states can be written as the wedge product of states of the form (4.2) with antiholomorphic differential forms on \mathcal{M}_k^0 . The quantum Hamiltonian is given by $H = H_{\text{COM}} + H^0$ where H_{COM} is the free centre of mass Hamiltonian, up to some factors of the form (4.1) (no factor of 4π), and H^0 contains the non-trivial dynamics and is given by (4.4) acting on \mathcal{M}_k^0 .

For the quantisation on $\mathcal{M}_1 \times \mathcal{M}_k^0$ to be valid on \mathcal{M}_k one needs to restrict the states to ensure that they are well defined differential forms on \mathcal{M}_k : the differential form on \mathcal{M}_k^0 must satisfy a discrete condition depending on the total electric charge of the state. For example, for $k = 2$ we can choose co-ordinates $(\mathbf{X}, \chi, r, \theta, \phi, \psi)$ on \mathcal{M}_2 where (θ, ϕ, ψ) are Euler angles and the action of Z_2 implies that we identify the point $(\mathbf{X}, \chi, r, \theta, \phi, \psi)$ with $(\mathbf{X}, \chi + \pi, r, \theta, \phi, \psi + \pi)$ (see [5]). Let Ψ be a state of the form (4.2) with $f_i(X) = \exp(i/\hbar(\mathbf{P}\mathbf{X} + S\chi))$ where the integer S is the total electric charge and Φ be an antiholomorphic form on \mathcal{M}_2^0 . For the state $\Psi \wedge \Phi$ to be well defined on \mathcal{M}_2 the antiholomorphic form on \mathcal{M}_2^0 must satisfy $\Phi(r, \theta, \phi, \psi + \pi) = (-1)^S \Phi(r, \theta, \phi, \psi)$. Such a restriction on the forms on \mathcal{M}_k^0 is to be implicitly understood in the following discussion.

The scattering theory can be investigated by constructing non-normalisable eigenforms of the Laplacian on \mathcal{M}_k^0 . In the asymptotic region a decomposition of the form (4.5) would allow the identification of these states with well separated monopoles and dyons in the spectrum. For \mathcal{M}_2^0 , the only case where the moduli space has been explicitly constructed, this seems a difficult undertaking.

The bound states are determined by the normalisable eigenforms on \mathcal{M}_k^0 . Since these states are tensored with a state of the form (4.2), this means that any bound state is actually part of a supermultiplet of bound states. Bound states of the bosonic theory, discussed in [5], automatically become part of a multiplet of bound states in the supersymmetric theory, corresponding to zero forms and $4k - 4$ forms on \mathcal{M}_k^0 . In general there will be a whole slew of bound states. It seems likely that the lowest energy bound state, for a given total electric charge, will be a true bound state of the full quantum field theory, since there isn't anything it could decay into.

We can be more explicit about a particularly important class of bound states. Fixing the total electric charge the energy of a bound state is bounded below by $4\pi k$ plus the energy coming from the centre of mass contribution. For a state to attain this bound, from (4.4) we see that the differential form on \mathcal{M}_k^0 must be annihilated by both the exterior derivative d and its adjoint d^\dagger . Such anti-holomorphic forms are equivalent to the dolbeault cohomology classes of \mathcal{M}_k^0 . Since there are no other bound states with lower energy it is plausible that these normalisable cohomology classes⁴ correspond to bound states in the full quantum field theory. This interpretation would be strengthened if we could show that the energy of the states is strictly less than the continuum (as opposed to being the lowest energy state bounding the continuum). Unfortunately it is not known to the author whether such normalisable harmonic forms exist on \mathcal{M}_k^0 ; perhaps it is not so difficult to determine if they exist on the Atiyah-Hitchin space \mathcal{M}_2^0 .

5. Discussion

We have shown that the dynamics of k monopoles of $N = 2$ supersymmetric Yang-Mills theory are determined by an effective $N = 4$ supersymmetric quantum mechanics based on the moduli space of k monopoles. The four worldline supersymmetries come from the fact that half of the supersymmetries in the field theory (described by four real parameters) are unbroken by a monopole configuration. We discussed how the Hilbert space of states of the effective theory are given by holomorphic differential forms on the moduli space and how the normalisable dolbeault cohomology classes of \mathcal{M}_k^0 are related to bound states of the full quantum field theory. It would be interesting to investigate the existence of such classes. The scattering problem requires the construction of eigenforms of the Laplacian on moduli space. Since the metric on moduli space is explicitly known for the case of two monopoles, perhaps some further progress could be made on these topics.

It would also be interesting to generalise the results of this paper to study the dynamics of the monopoles of $N = 4$ supersymmetric Yang-Mills. Although there appear to be some technical complications in repeating the steps in this paper the result seems almost certain. A monopole configuration will still only break half of the supersymmetries of the field theory. Since there are now twice as many supersymmetries the unbroken supersymmetries are determined by eight real parameters instead of four. We thus expect the low energy dynamics to be described by an $N = 8$ supersymmetric quantum mechanics (eight real parameters) based on the moduli space \mathcal{M}_k . In addition the Callias index theorem says there will be twice as many fermionic zero modes as in the $N = 2$ model. Putting this together we are led to an effective action of the form

$$S = \frac{1}{2} \int dt \left[\dot{X}^a \dot{X}^b \mathcal{G}_{ab} + i \bar{\psi}^a \gamma^0 D_t \psi^b \mathcal{G}_{ab} + \frac{1}{6} \mathcal{R}_{abcd} (\bar{\psi}^a \psi^c) (\bar{\psi}^b \psi^d) \right] \quad (5.1)$$

where ψ^a is a two component real spinor (i.e. two worldline Grassmann numbers), $\gamma^0 = \sigma^2$ and \mathcal{R}_{abcd} is the Riemman tensor of \mathcal{G} . This action is the dimensional reduction of the

⁴ If we had described the Hilbert space using spinors on moduli space, these bound states would be given by zero modes of the Dirac operator on \mathcal{M}_k^0 .

two dimensional supersymmetric sigma model and it is well known that when the metric is hyperKähler it admits four supersymmetries, each parametrised by a two component Majorana spinor [22]. Thus (5.1) based on \mathcal{M}_k does indeed admit eight worldline supersymmetries (it is a matter of convention to describe this action as having $N = 4$ or $N = 8$ worldline supersymmetry; we stick to the nomenclature that N equals the number of real worldline supersymmetry parameters). The quantisation of this model leads to a Hilbert space of states consisting of real differential forms on the moduli space [23]. In this case the real normalisable harmonic forms of \mathcal{M}_k^0 will be related to bound states of the full quantum field theory.

An interesting feature of the $N = 4$ model is that the monopole spectrum contains particles of spin 1 [21]. Consequently, one might expect the amplitudes for the scattering of such monopoles to obey Ward identities. It would be very interesting if such identities were encoded in the geometry of moduli space. Another reason for pursuing more detailed calculations for this model is that it may be possible to gain some insight into the Montonen and Olive duality conjecture [24],[21]. In particular this conjecture says that low energy scattering of monopoles at weak coupling should correspond to low energy scattering of elementary particles at strong coupling. We can possibly make some progress on the monopole scattering but a direct comparison with the elementary particle scattering is not possible because it is at strong coupling. Nevertheless, the discovery of Ward identities mentioned above would possibly be further evidence that the conjecture is correct.

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